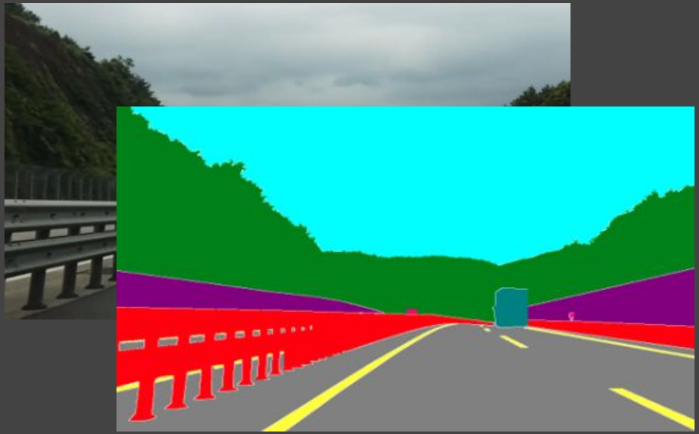
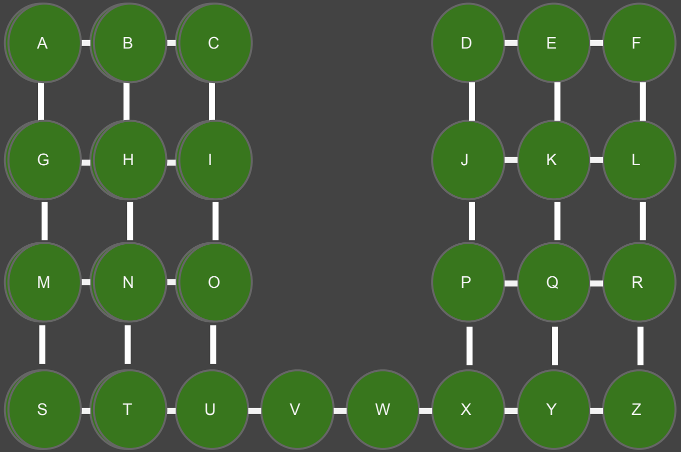
Probabilistic Graphical Models

Lecture 6

Markov Random Fields

What about ….



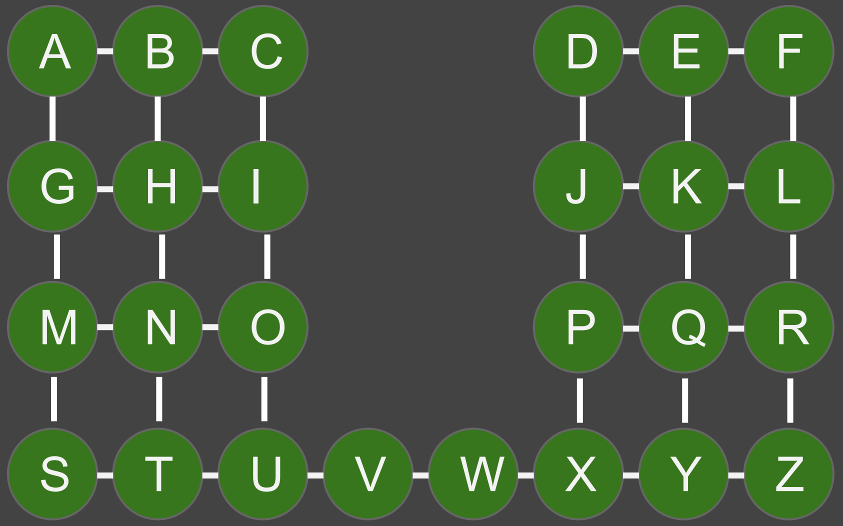


Influenza Transmission in Class

$X_A \in \{0,1\}$: student 'A' has flu
$X_B \in \{0,1\}$: student 'B' has flu
⋮
$X_Z \in \{0,1\}$: student 'Z' has flu

A can only transmit to B,G

H can only transmit to B,G,I,N



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Is X_A independent of X_Z?

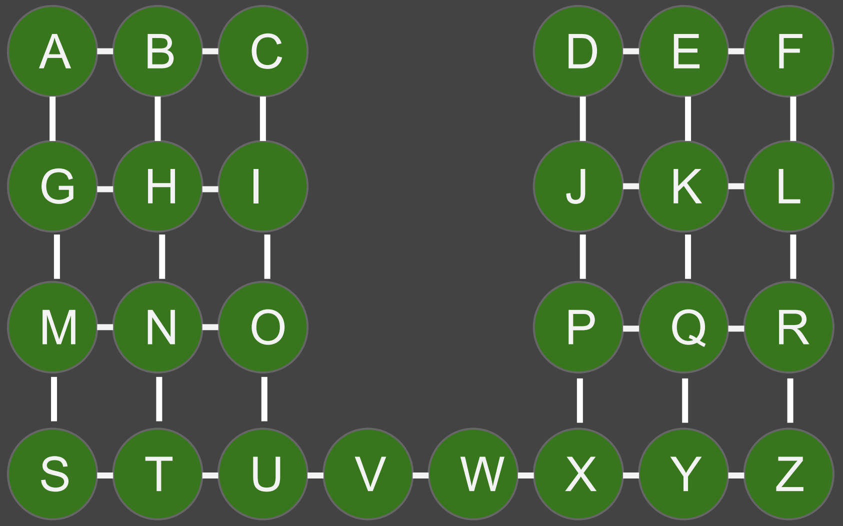
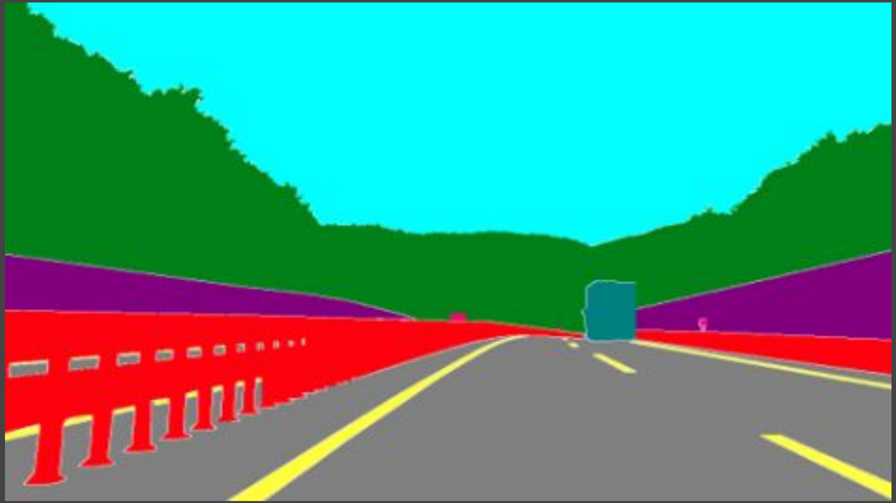


Image segmentation

$X_i \in$ {Road, Lanemark, Sky, Vegetation, Guardrail, ...}

X_i: Label of the node at pixel 'i'



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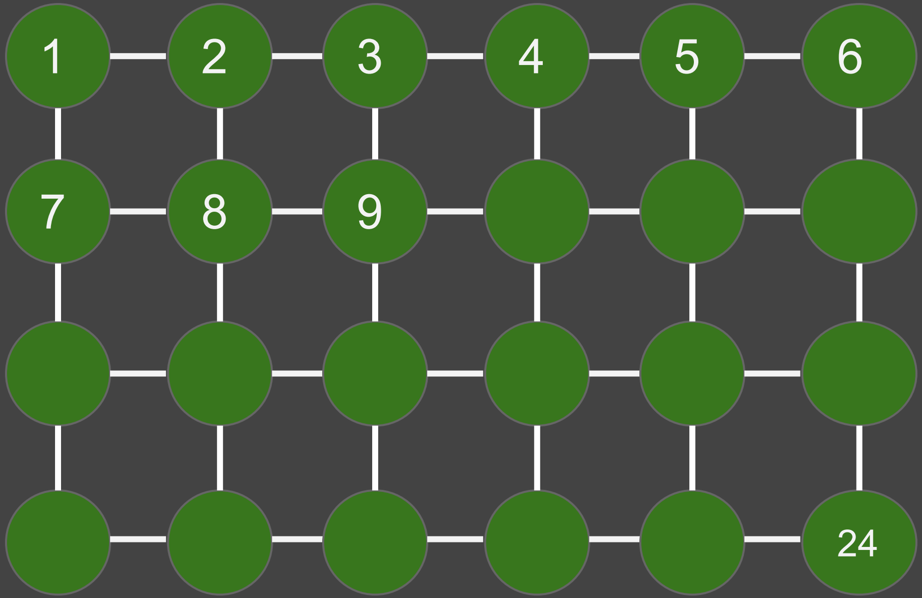

Image segmentation

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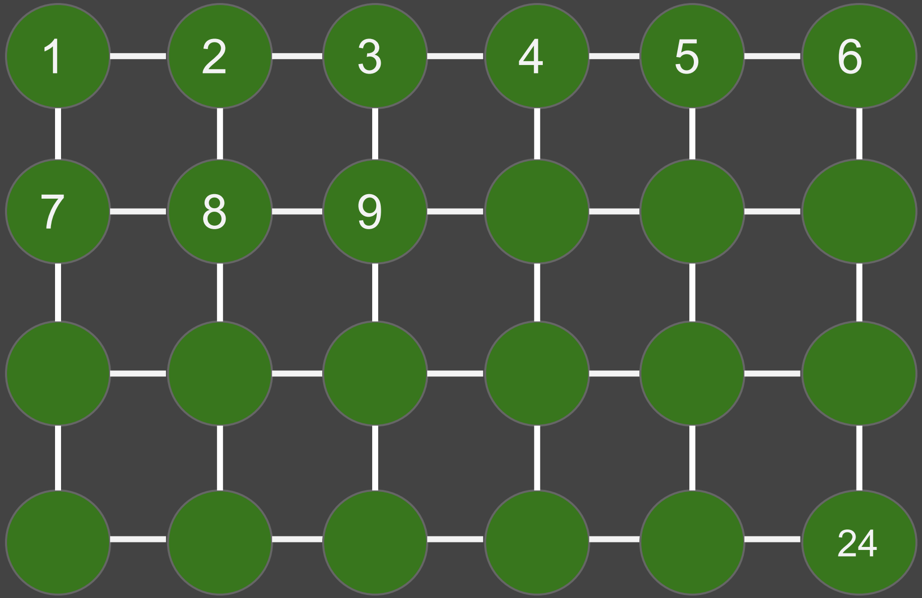
$X_i \in$ {Road, Lanemark, Sky, Vegetation, Guardrail, …}

X_i: Label of the node at pixel 'i'

Is X_1 independent of X_{24}?



Influenza Transmission in Class

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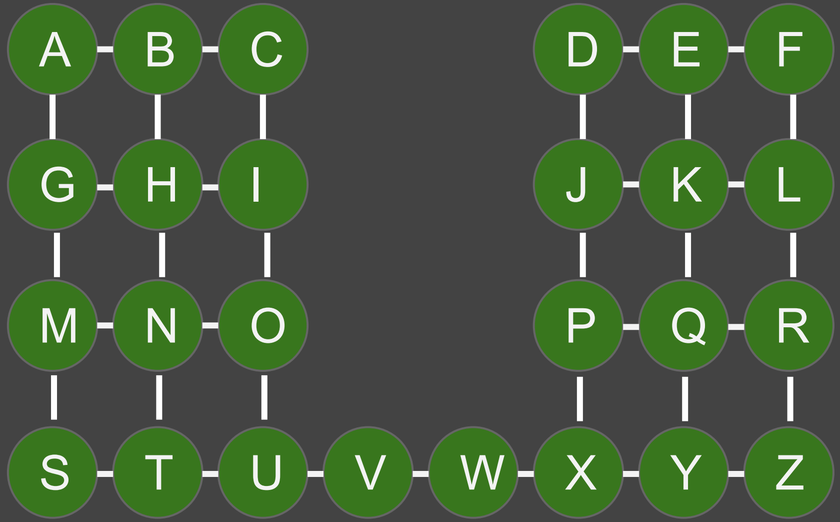

Is X_A independent of X_Z? What sort of independent do we have here?

Influenza Transmission in Class

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$X_B \in \{0,1\}$: student 'B' has flu

⋮

$X_Z \in \{0,1\}$: student 'Z' has flu

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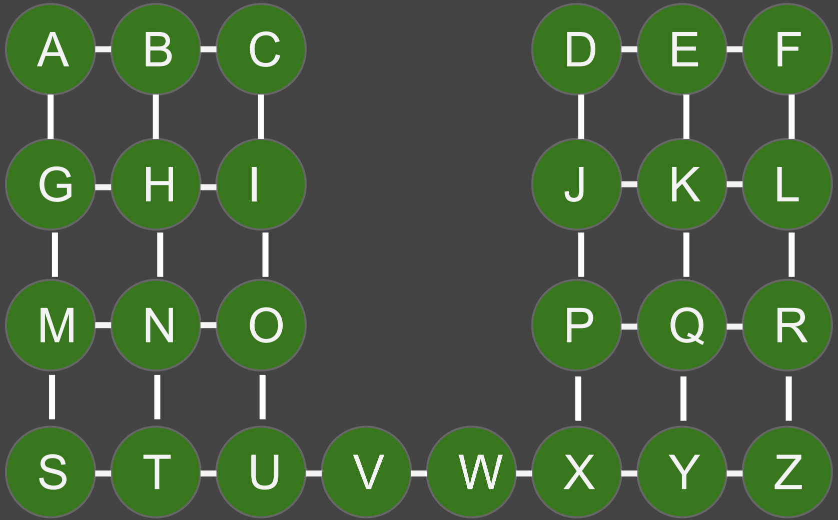

Given X_B and X_G, is X_A independent of X_Z?

Influenza Transmission in Class

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$X_A \in \{0,1\}$: student 'A' has flu
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A can only transmit to B,G

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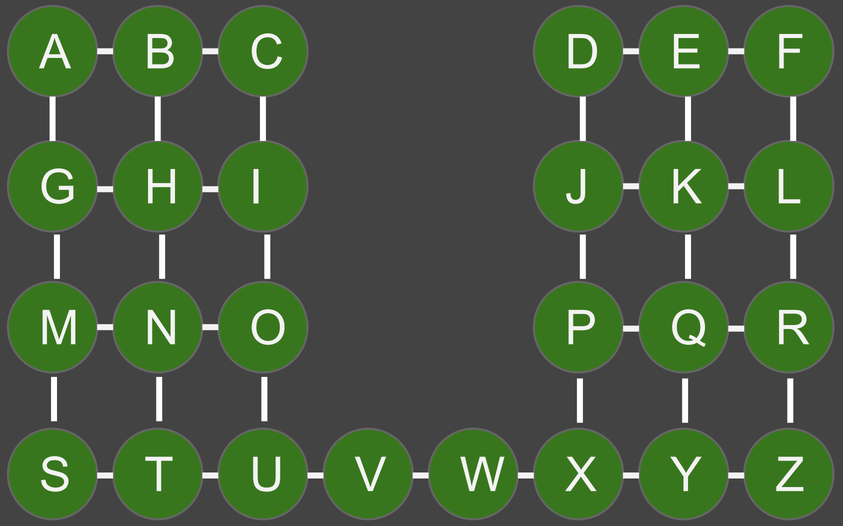
$P(X_A \mid X_Z, X_B, X_G) = ?$



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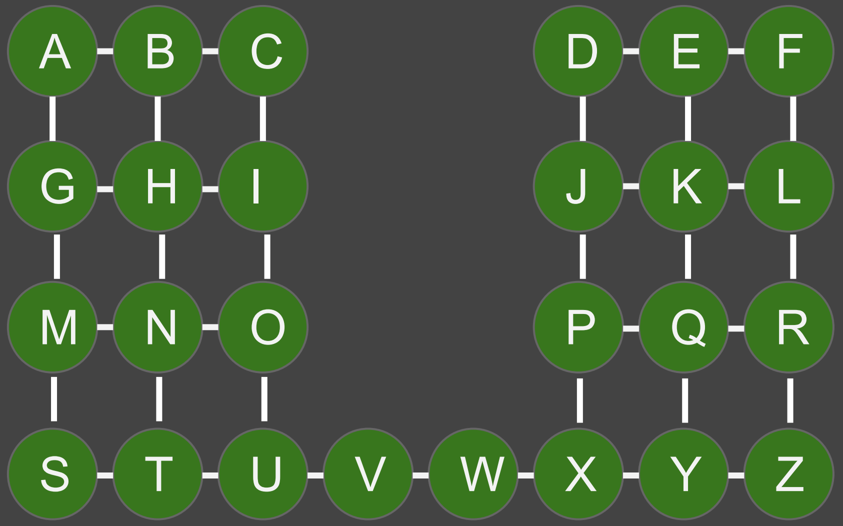

A can only transmit to B,G

H can only transmit to B,G,I,N

$$P(X_A \mid X_Z, X_B, X_G) = P(X_A \mid X_B, X_G)$$

Influenza Transmission in Class

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A can only transmit to B,G

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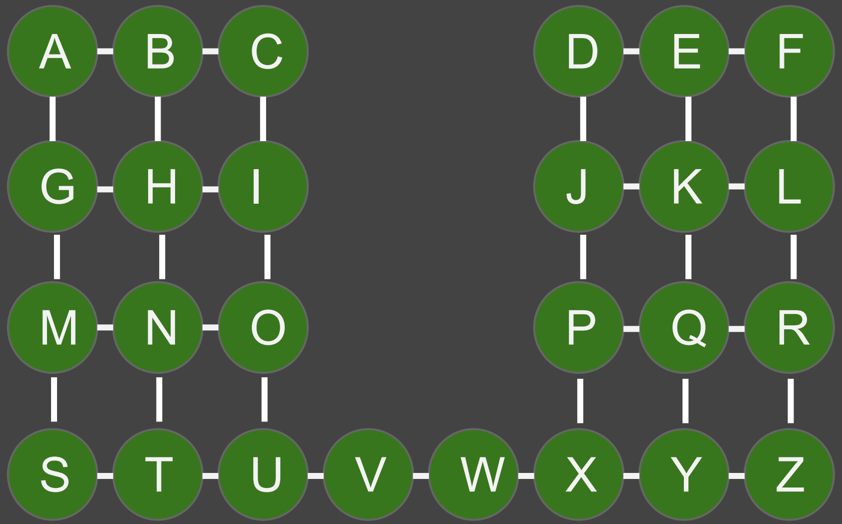
$P(X_A \mid X_B, X_C, X_D, \ldots, X_Y, X_Z) = ?$



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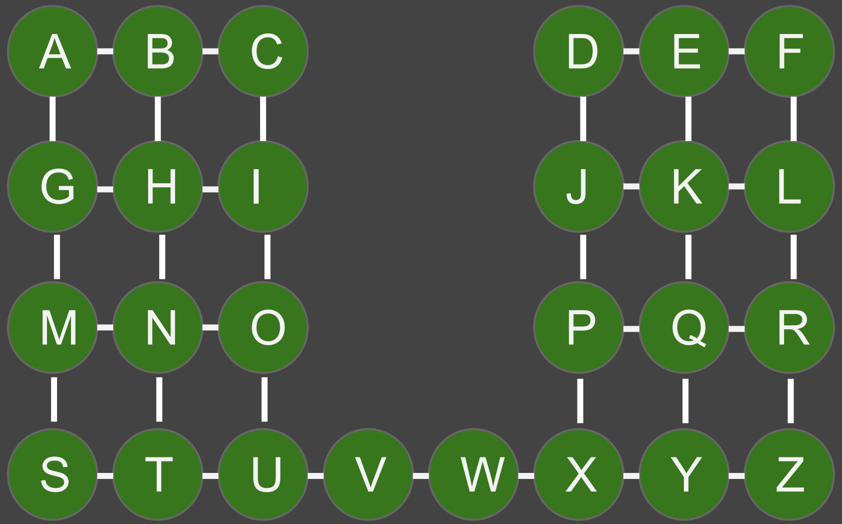
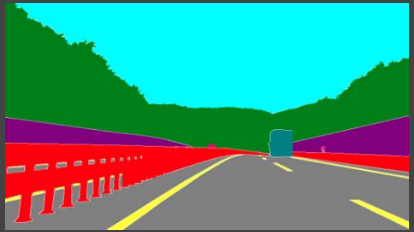


$$P(X_A \mid X_B, X_C, X_D, \ldots, X_Y, X_Z) = P(X_A \mid X_B, X_G)$$

Image segmentation



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$X_i \in$ {Road, Lanemark, Sky, Vegetation, Guardrail, ...}

X_i: Label of the node at pixel 'i'

Is X_1 independent of X_{24}?

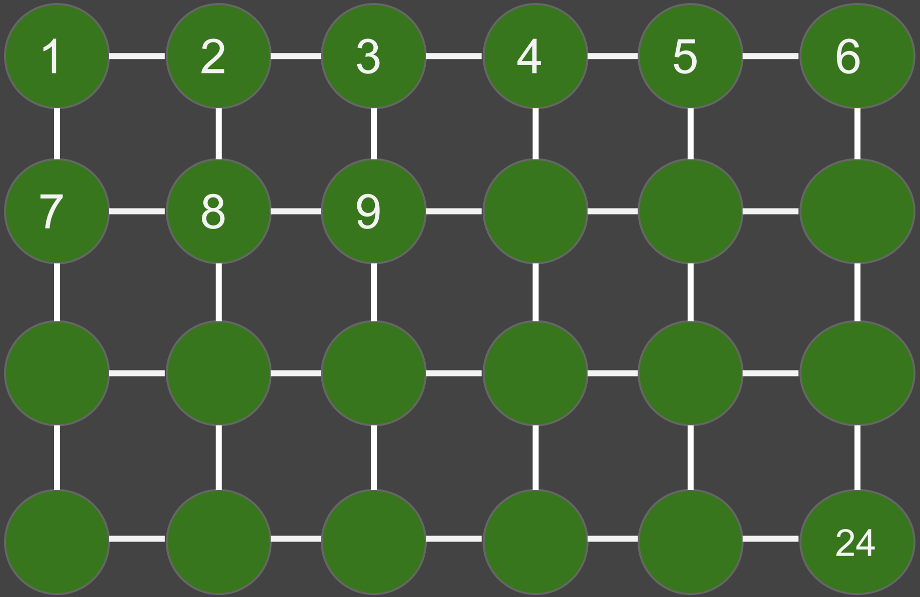
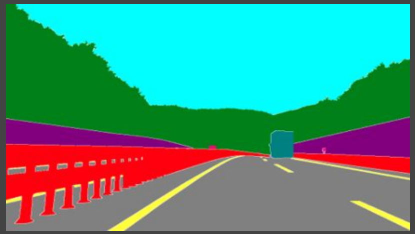


Image segmentation



$X_i \in$ {Road, Lanemark, Sky, Vegetation, Guardrail, ...}

X_i: Label of the node at pixel 'i'

Given X_2, X_7 and X_8, is X_1 independent of X_{24}?

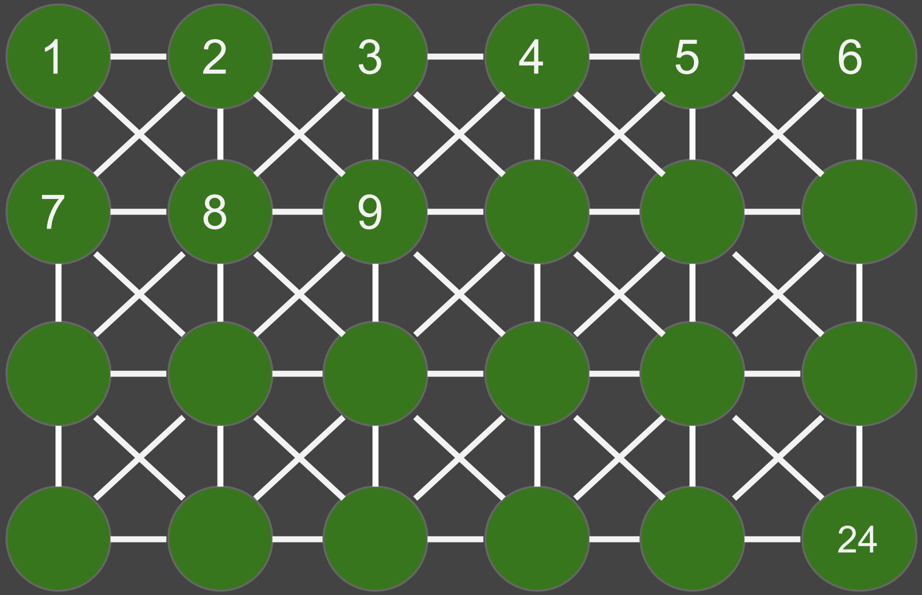
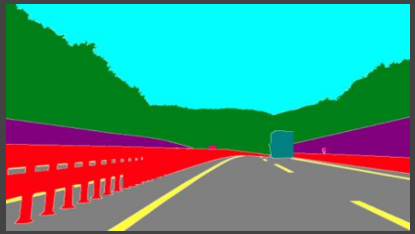


Image segmentation



$X_i \in$ {Road, Lanemark, Sky, Vegetation, Guardrail, ...}

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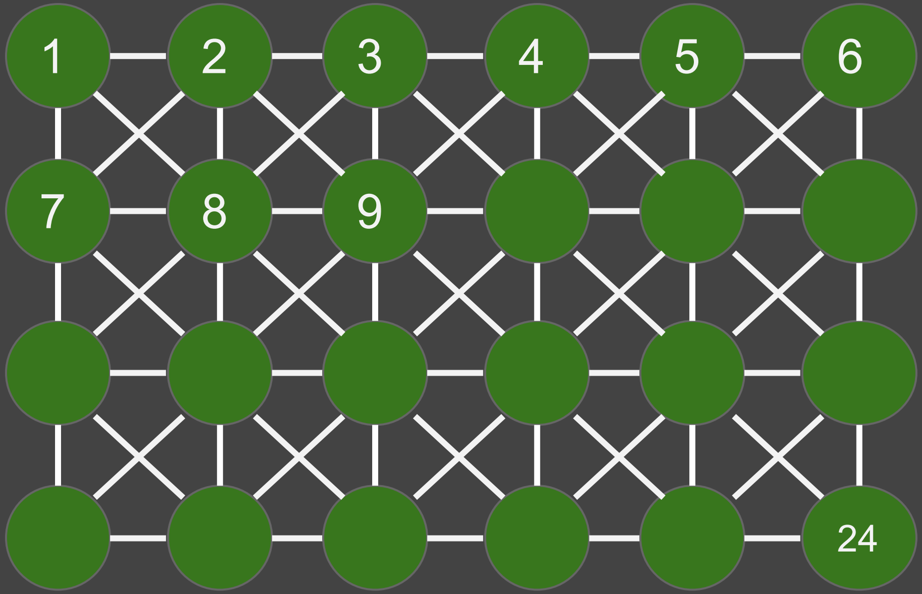
Given X_2, X_7 and X_8, is X_1 independent of X_{24}? Maybe!



Markov Random Field

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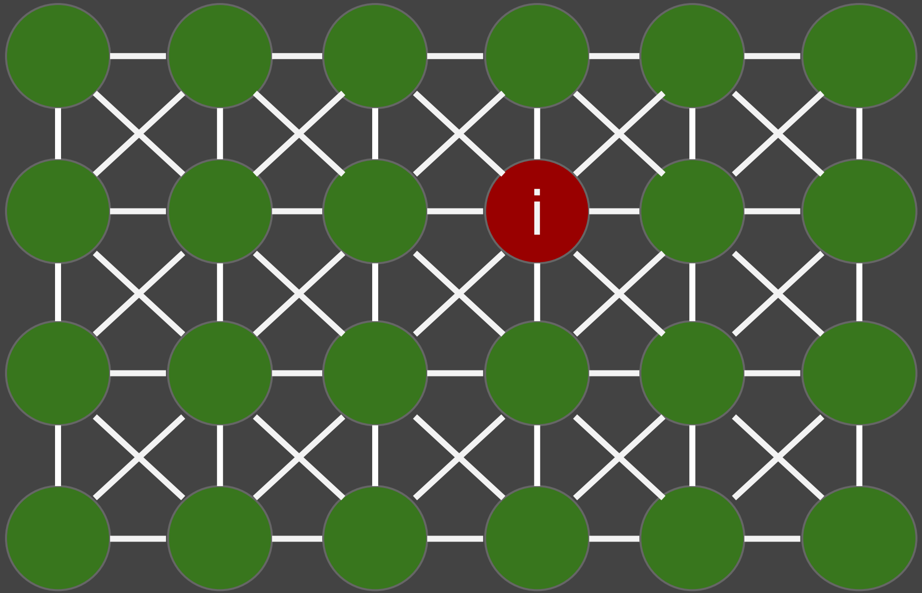
I = set of all nodes

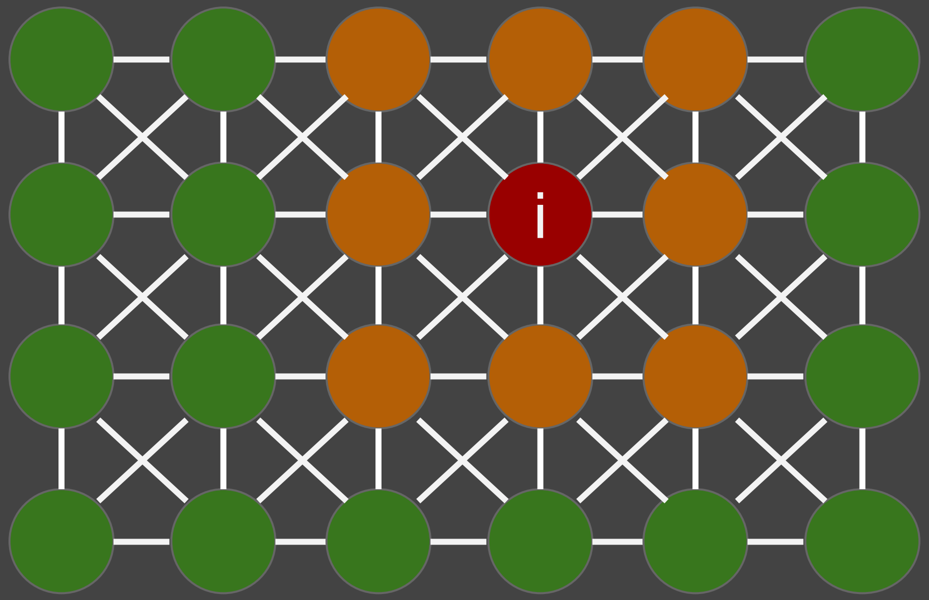
$P(X_i \mid X_{I\setminus\{i\}}) =$



Markov Random Field (MRF)

I = set of all nodes

$P(X_i \mid X_{I \setminus \{i\}}) = P(X_i \mid X_{N(i)})$

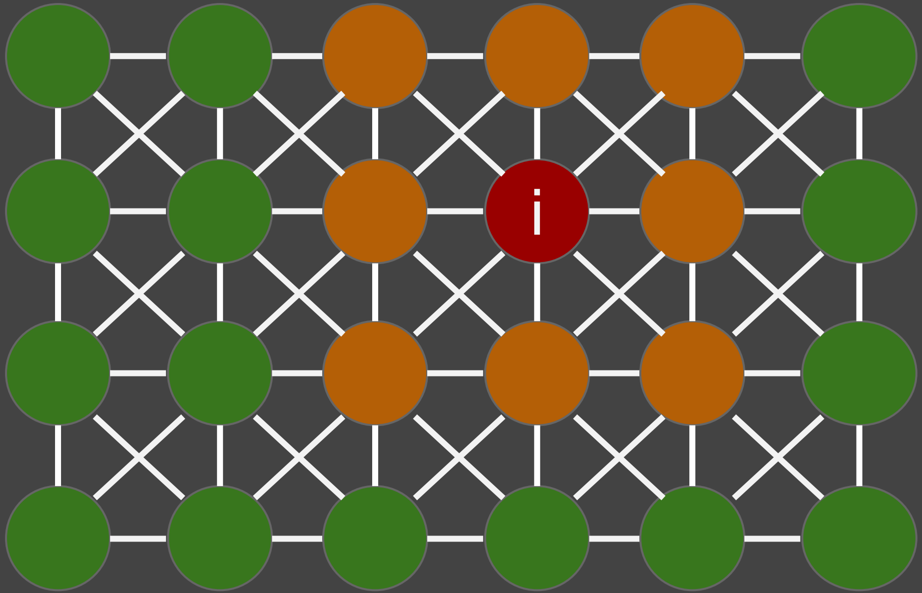


Markov Random Field

I = set of all nodes

$P(X_i \mid X_{I \setminus \{i\}}) = P(X_i \mid X_{N(i)})$

N(i): neighbours of node i



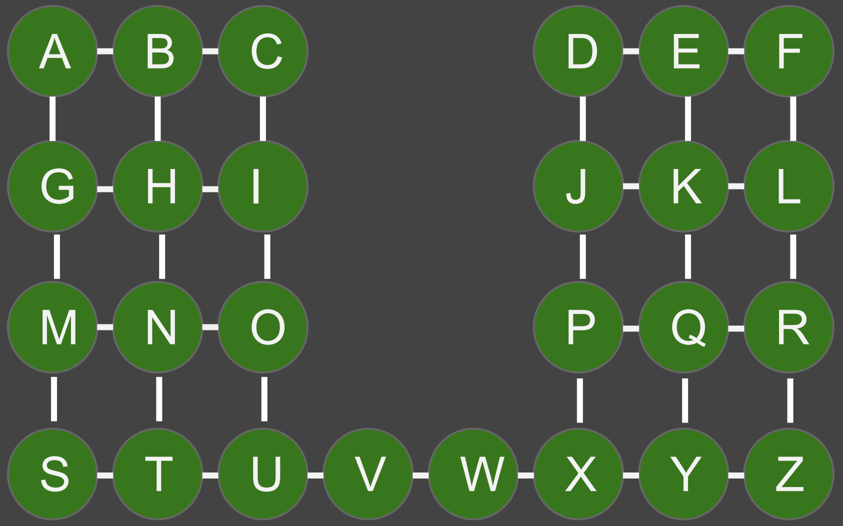
Gibbs Random Fields (Gibbs Distribution)

$P(X_A, X_B, X_C, X_D, \ldots, X_y, X_Z) =$

$1/Z \;\; f_1(X_A)\, f_2(X_B) \ldots f_{26}(X_Z)$

$g_1(X_A, X_B)\;\; g_2(X_B, X_C)\;\; g_3(X_A, X_G)$
$\ldots\; g_{36}(X_R, X_Z)\;\; g_{37}(X_y, X_Z)$

$f_i(x) > 0,\; g_j(x,y) > 0$ for all x,y



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Gibbs random fields

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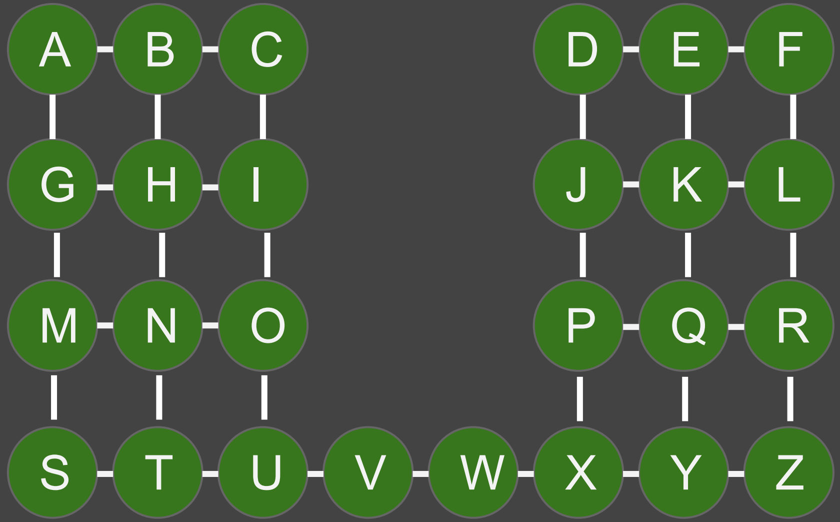
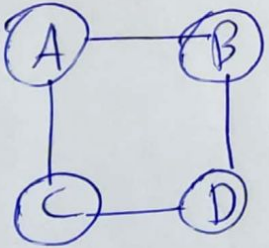
$\ldots g_{36}(X_R, X_Z) \; g_{37}(X_Y, X_Z)$

$f_i(x) > 0, g_j(x,y) > 0$ for all x,y

Z: the partition function



Gibbs random fields



$$p(A,B,C,D) = f_1(A)\, f_2(B)\, f_3(C)\, f_4(D)$$

$$g_{12} \quad f_{12}(A,B)\, f_{24}(B,D)$$

$$f_{13}(A,C)\, f_{34}(C,D)$$

$$g_{13}$$

$$= \frac{g_{12}(A,B)\, g_{24}(B,D)}{g_{13}(A,C)\, g_{34}(C,D)}$$

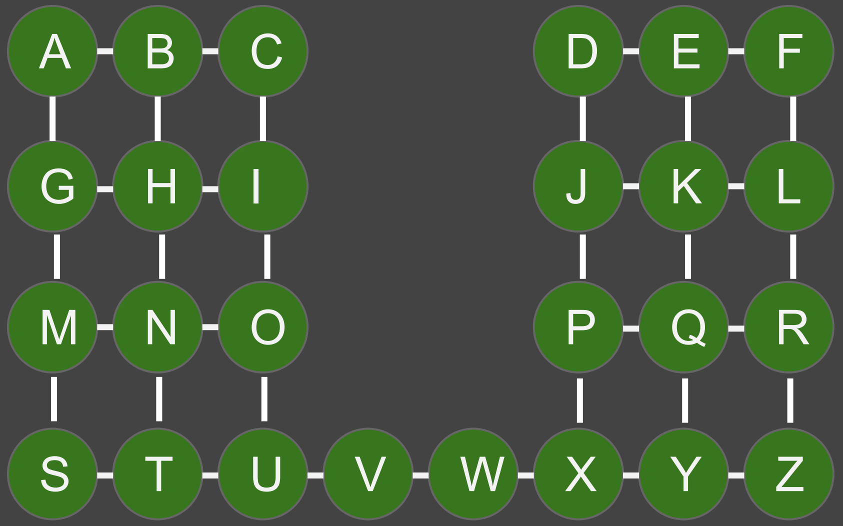
Gibbs random fields

$P(X_A, X_B, X_C, X_D, \ldots, X_Y, X_Z) =$

$1/Z \; g_1(X_A, X_B) \; g_2(X_B, X_C) \; g_3(X_A, X_G) \ldots g_{36}(X_R, X_Z) \; g_{37}(X_Y, X_Z)$

$g_j(x,y) > 0$ for all x,y

Z: the **partition function**



Gibbs random fields

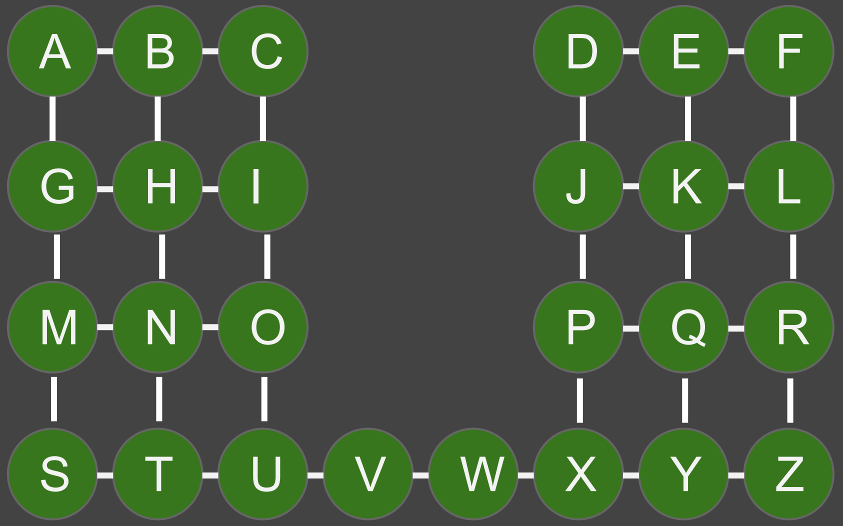
$P(X_A, X_B, X_C, X_D, \ldots, X_Y, X_Z) =$

$1/Z \;\; g_1(X_A, X_B) \;\; g_2(X_B, X_C)$
$g_3(X_A, X_G) \ldots g_{36}(X_R, X_Z) \;\; g_{37}(X_Y, X_Z)$

$g_j(x,y) > 0$ for all x,y

Z: the **partition function**

what is Z?



Generalized Gibbs Distribution

$P(X_A, X_B, X_C, X_D, \ldots, X_y, X_Z) =$

$1/Z\ \ f_1(X_A)\, f_2(X_B) \ldots f_{26}(X_Z)$

$g_1(X_A, X_B)\ \ g_2(X_B, X_C)\ \ g_3(X_A, X_G)$
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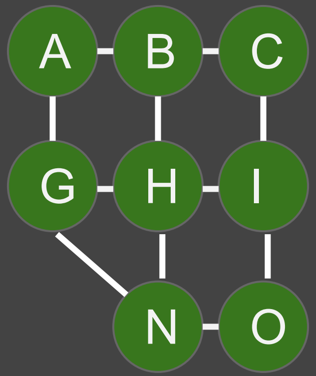
what is Z?

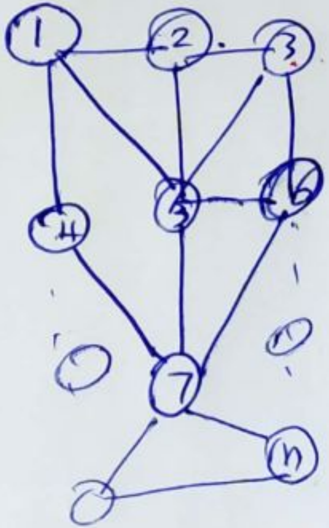


Generalized Gibbs Distribution

$$P(X_1, X_2, \dots, X_n) = \frac{1}{Z} \underbrace{\prod_{i=1}^{n} f_i(X_i) \prod_{(i,j)\in\mathcal{E}} f_{ij}(X_i, X_j)}$$

$\tilde{P}(X_1, X_2, \dots, X_n)$

unnormalized measure



$(\mathcal{V}, \mathcal{E})$

$\mathcal{V} = \{1, 2, \dots, n\}$

$\mathcal{E} = \{(1,2), (2,3), \dots, (7,n)\}$

Generalized Gibbs Distribution

$$\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} P(x_1, x_2, \dots, x_n) = 1$$

$$\Rightarrow \sum_{x_1, x_2, \dots x_n} \frac{1}{Z} \prod_{i=1}^{n} f_i(x_i) \prod_{(i,j)\in E} f_{ij}(x_i, x_j) = 1$$

$$Z = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} \prod_{i=1}^{n} f_i(x_i) \prod_{(i,j)\in E} f_{ij}(x_i, x_j)$$

$$Z = \sum_{x_1}\sum_{x_2}\sum_{x_n} \tilde{P}(x)$$

Generalized Gibbs Distribution

$$X = (X_1, X_2, X_3, \dots, X_n)$$

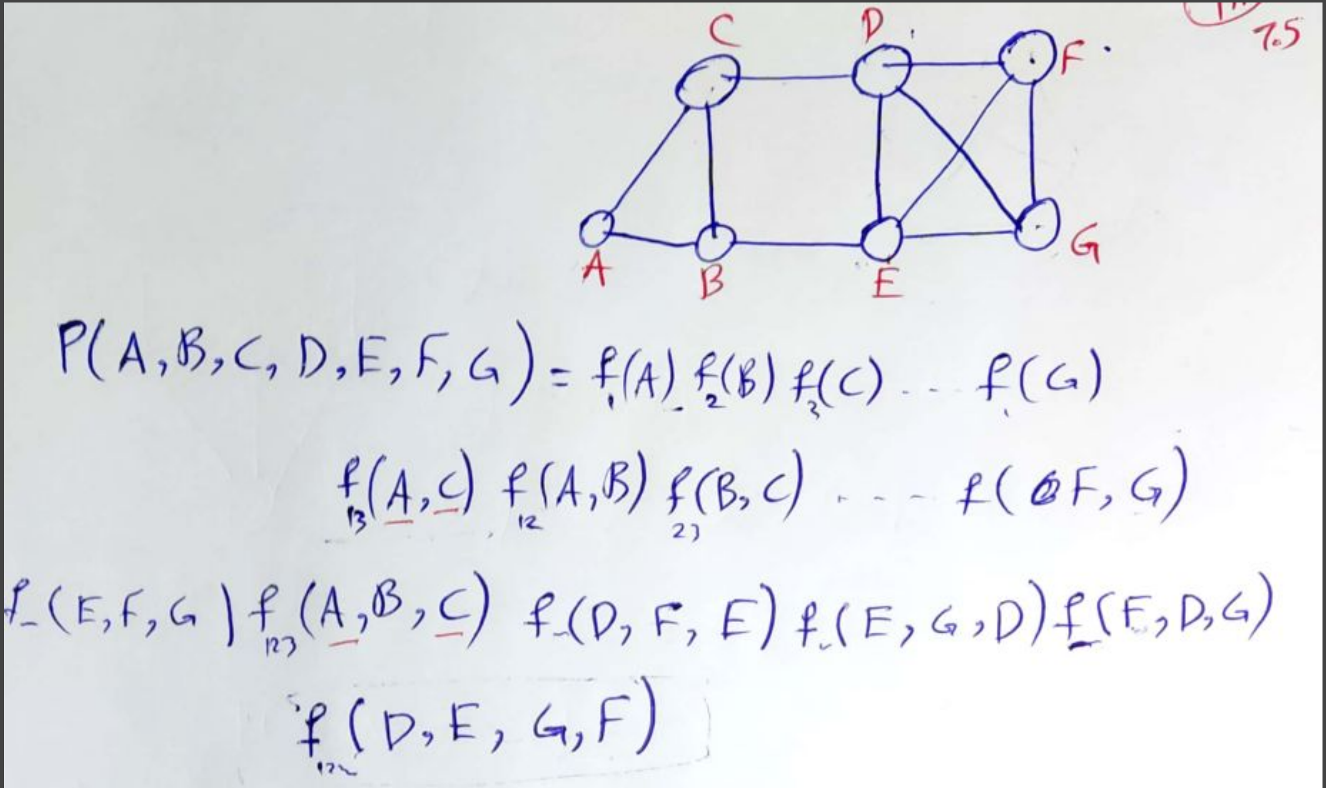
$$P(X_1, X_2, \dots, X_n) = P(X)$$

$$\tilde{P}(X) = \prod_{i=1}^{n} f_i(X_i) \prod_{(i,j) \in \mathcal{E}} f_{ij}(X_i, X_j)$$

$$P(X) = \frac{1}{Z} \tilde{P}(X)$$

$$Z = \sum_{X} \tilde{P}(X)$$

Generalized Gibbs Distribution



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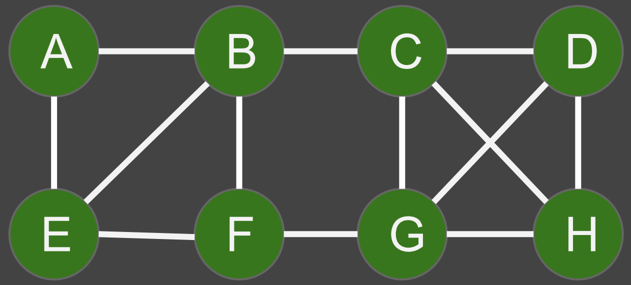
Generalized Gibbs Distribution

(X,Y) appear in a factor => there is an edge between X and Y in graph

Cliques

Fully connected subgraphs



Cliques

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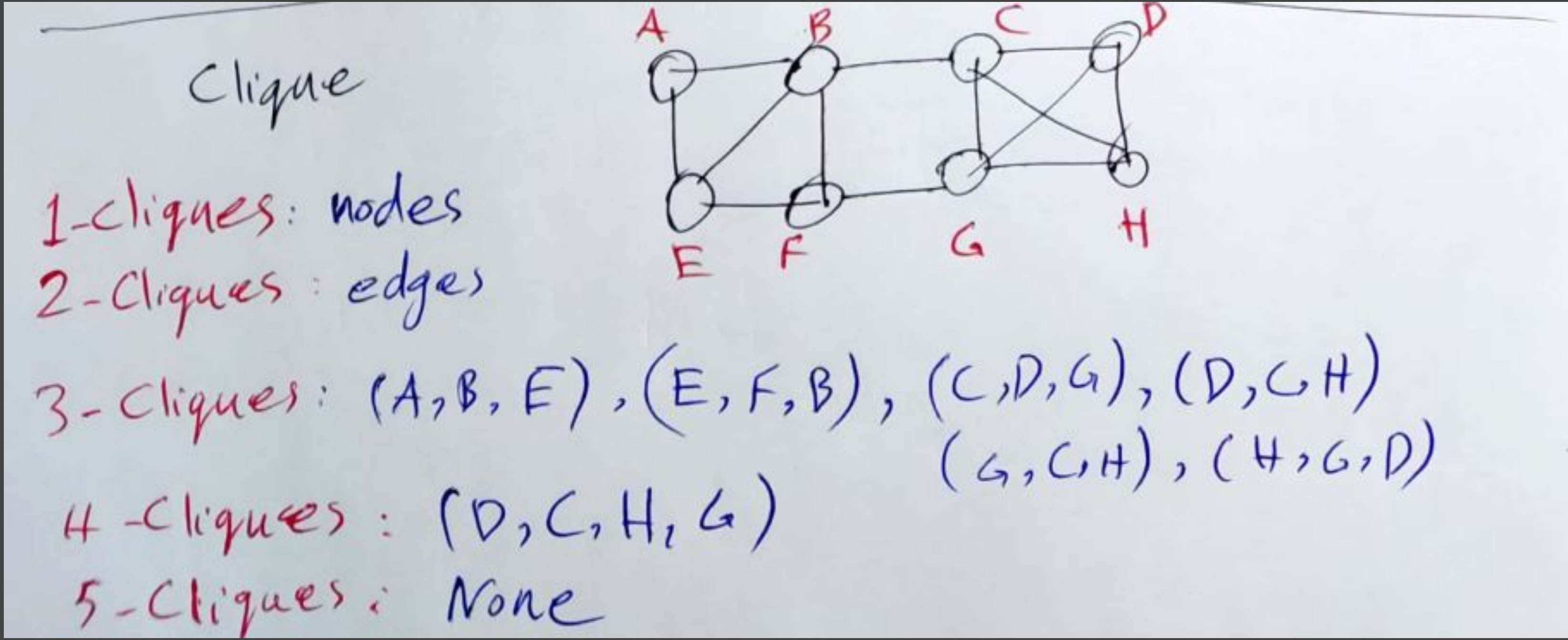

Cliques

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Cambridge Dictionary

clique

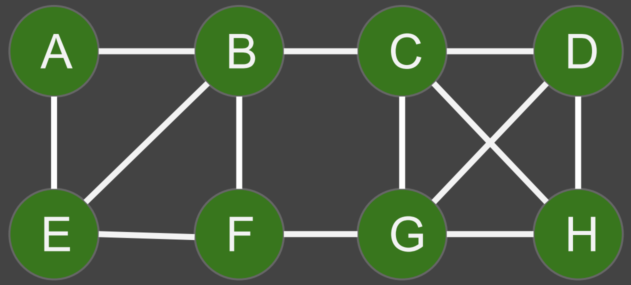
noun [C, + sing/pl verb] • disapproving

a small group of people who spend their time together and do not welcome other people into that group:

Cliques

maximal cliques: cliques not contined in larger cliques

(A,B,E), (B,F,E), (B,C), (F,G), (C,D,G,H)



Two random fields

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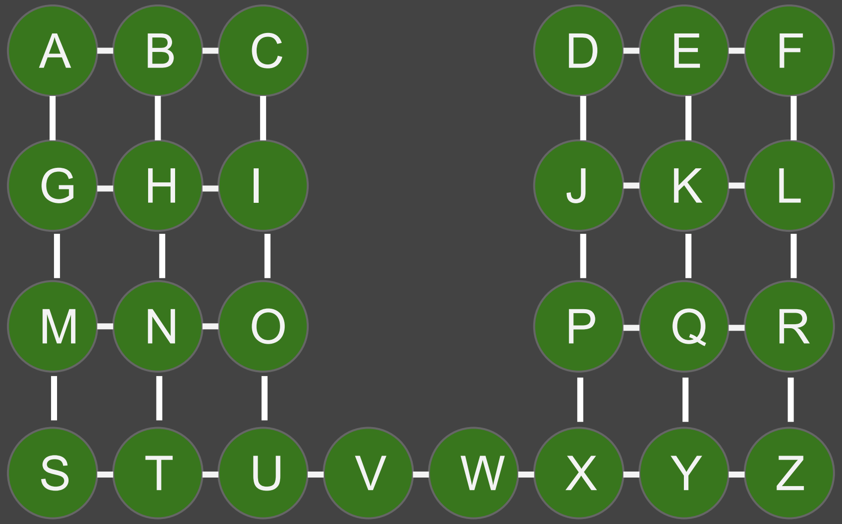
Markov random field:

$$P(X_i \mid X_{I\setminus\{i\}}) = P(X_i \mid X_{N(i)})$$

Gibbs random field:

$$P(X_A, X_B, X_C, X_D, \ldots, X_y, X_Z) =$$

$$1/Z \;\; g_1(X_A, X_B) \;\; g_2(X_B, X_C) \;\; g_3(X_A, X_G) \ldots g_{36}(X_R, X_Z) \;\; g_{37}(X_y, X_Z)$$



Generalized Gibbs Distribution

Generalized Gibbs Distribution

IV 7.5

$P(X_1, X_2, \ldots, X_n)$

Joint distribution

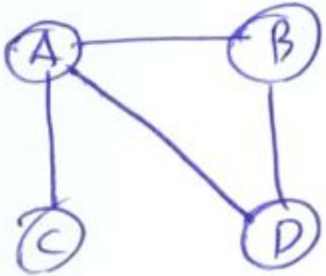
$G = (V, \mathcal{E})$

Undirected Graph

$P(X_1 — X_n)$ can be written as ~~the~~ a product of factors s.t. each factor is a function of a clique of G.

Generalized Gibbs Distribution

$$P_1(A,B,C,D) = f(A,C)\,f(A,D)\,f(B,D)\,f(A,B)$$



G

$$P_2(A,B,C,D) = f(A,C)\,f(A,D,B)$$

P_1, P_2 are both Gibbs distributions corresponding to G.

Hammersley-Clifford theorem

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Assume that $P(X_1, X_2, X_3, \ldots, X_n) > 0$ for all assignments to $X_1, X_2, \ldots, X_n$, then

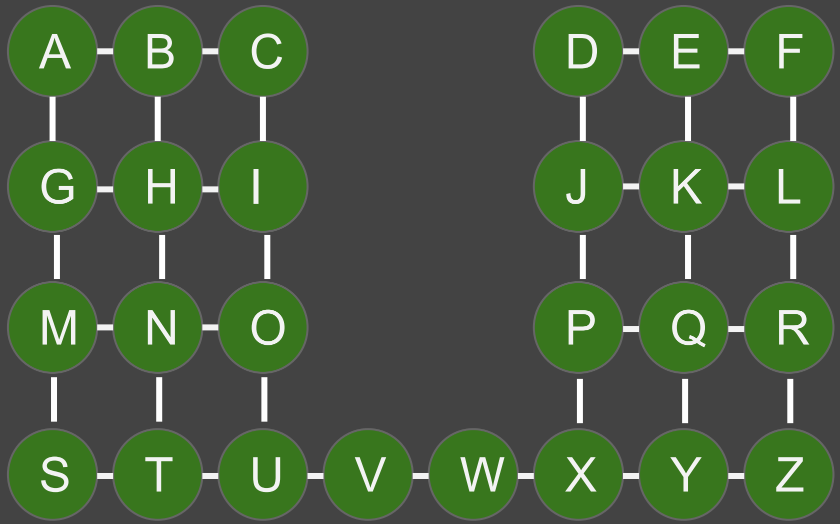
- P is a Markov Random Fields over an unidrected graph G, if and only if it is a Gibbs distribution (factorizes over the cliques of the graph)

Examples

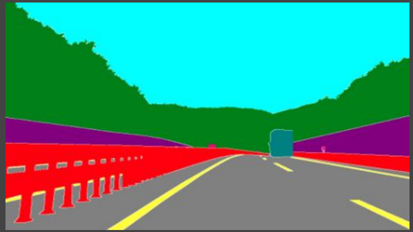
$P(X_A, X_B, X_C, X_D, \ldots, X_Y, X_Z) =$

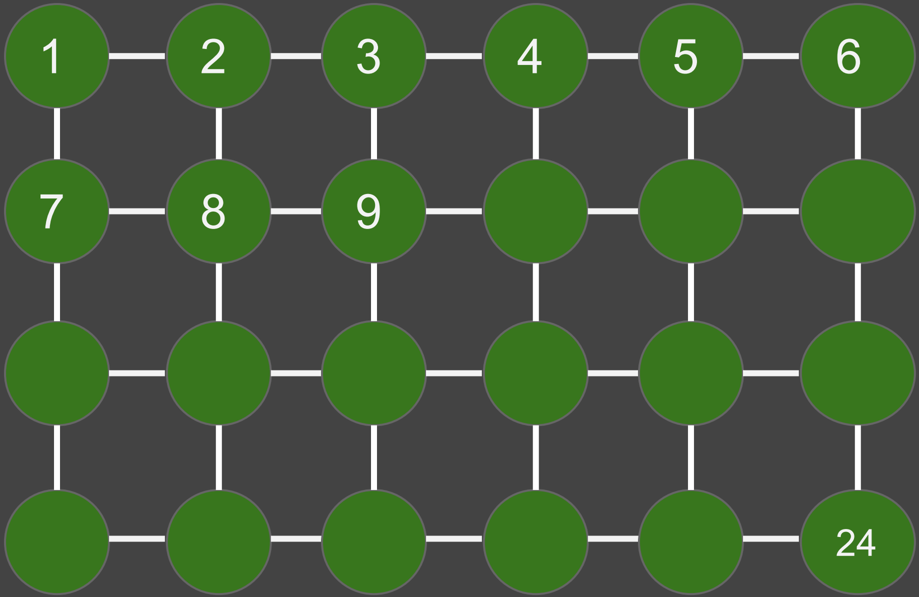
$1/Z \; f_1(X_A)\, f_2(X_B) \ldots f_{26}(X_Z)$

$g_1(X_A, X_B)\; g_2(X_B, X_C)\; g_3(X_A, X_G)$
$\ldots\; g_{36}(X_R, X_Z)\; g_{37}(X_Y, X_Z)$



Examples





Markov Property

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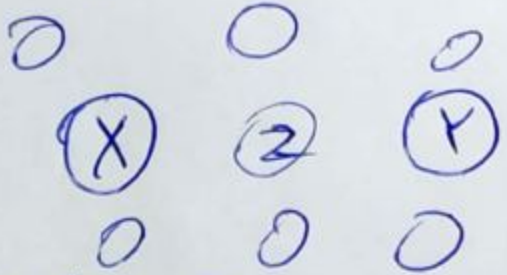
1. Pairwise Markov Property
2. **Local Markov Property** $P(X_i \mid X_{I \setminus \{i\}}) = P(X_i \mid X_{N(i)})$
3. Global Markov Property

All are true for Gibbs random fields corresponding to the cliques of the graph.

=> $P(X_1, X_2, X_3, \ldots, X_n) > 0$ 1,2,3 are equivalent

Pairwise Markov Property

~~X, Y pair~~ pairwise markov property



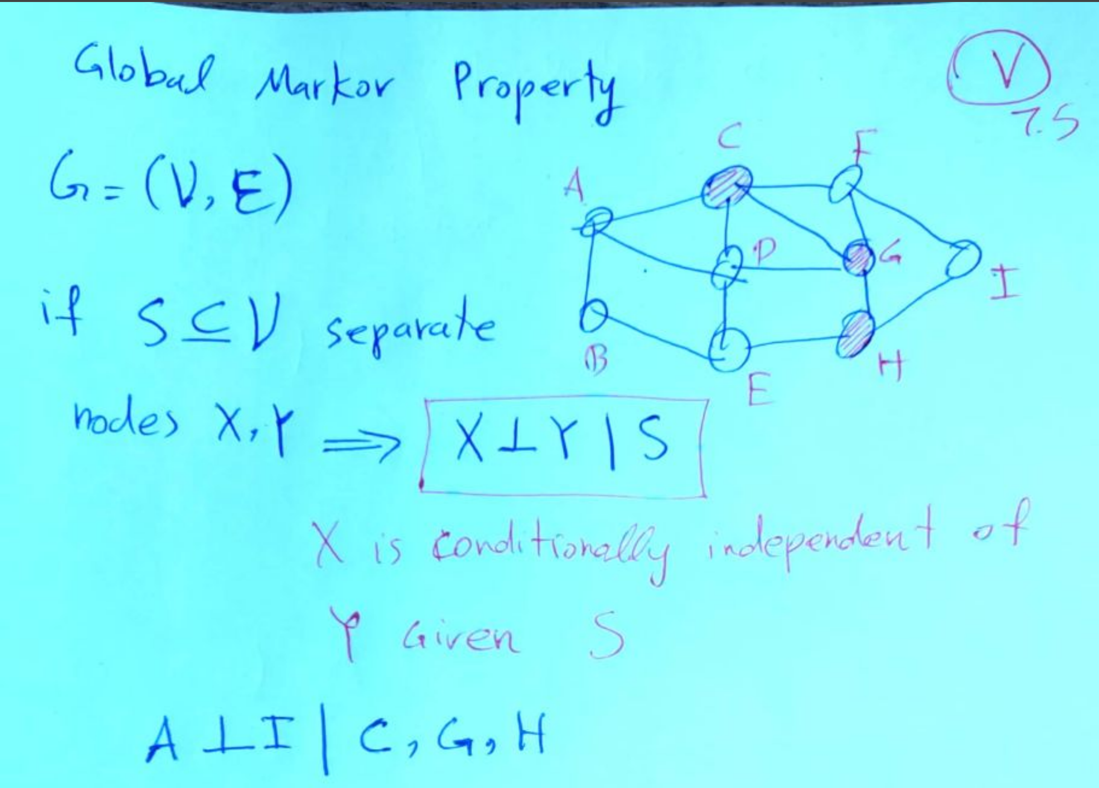
$G = (V, \mathcal{E})$

$(X, Y) \notin \mathcal{E}$

X, Y are independent given all other nodes

$X \perp Y \mid V \setminus \{X, Y\}$

Global Markov Property



Example